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http://www.staff.science.uu.nl/~dijks101/C2019
The Mathematics of Climate and the Environment

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A tutorial is provided on the application of dynamical systems theory to problems in climate dynamics. We start with the analysis low-dimensional deterministic dynamical systems using bifurcation theory and provide examples in conceptual climate models. We then proceed to stochastic low-dimensional systems and eventually end with operator based techniques within ergodic theory.

Videos (all lectures):
https://climate-houches.sciencesconf.org/resource/page/id/8
Dynamical Systems Approach

- Bifurcation Theory
- Ergodic Theory
- Trajectories in Phase space
- Attractors in Phase space
- Physical space
- Time Series

Bifurcation Theory

Ergodic theory
Artist’s view of Climate Variability

Mitchell, 1978
Climate variability on Greenland during the last 125,000 years

Strong indications of ocean circulation changes

Wind stress

(a) Mean wind stress and momentum flux 1984–2006 (N/m²)
Heat Flux

(b) Mean heat flux 1984–2006 (W/m²)
Fresh water flux

(c) Mean water flux 1984–2006 (cm/yr)
Surface Circulation
CESM results, UU 2018
(1 year daily data)

Sea Surface Height in centimeter

Date: 01-01-0044
Along Atlantic N-S section:

**Temperature**

Temperature (°C)

**Salinity**

Salinity (ppt)
Meridional Overturning Circulation (MOC)

\[ \Psi(y, z, t) = \int_0^0 \int_{x_w}^{x_e} v(x, y, z', t) \, dx \, dz' \]

\( v \): meridional velocity

Model determined

Meridional overturning streamfunction (Sv)

MOC: Volume transport in latitude/depth (is observable)
Observations of the MOC

RAPID-MOCHA array
Observations: MOC strength

https://www.rapid.ac.uk/rapidmoc/
Recent changes in the MOC

Smeet et al., Ocean Science, (2014)
Connection MOC and meridional heat transport

Johns et al., J Clim, (2010)
Global Climate Model (FAMOUS): MOC collapse

Hawkins et al., GRL, (2011)

freshwater flux (Sv)

freshwater increase
The salt-advection feedback

\[ \tau_T \quad \tau_S \]

- no damping of salinity anomalies
- strong damping of heat anomalies

Equator  \hspace{2cm} North
Tipping elements in the climate system

A component of the Earth system, at least sub-continental in scale (~1000km), that can be switched – under certain circumstances – into a qualitatively different state by a small perturbation.

Lenton et al., (2008)
Dynamical Systems Approach

- Bifurcation Theory
- Ergodic theory

Trajectories in Phase space

Attractors in Physical space

Time Series

Bifurcation Theory
Elementary bifurcations

Saddle-node (non-existence)

\[ \frac{dx}{dt} = \lambda - x^2 \]

Pitchfork (symmetry-breaking)

\[ \frac{dx}{dt} = \lambda x - x^3 \]

Transcritical (exchange of stability)

\[ \dot{x} = \lambda x - \omega y - x(x^2 + y^2) \]
\[ \dot{y} = \lambda y + \omega x - y(x^2 + y^2) \]

Hopf (spontaneous oscillations)
Ex: Hopf bifurcation
Why determine full bifurcation diagrams of autonomous models (B-tipping)?

1. Physical mechanisms

2. (Correct) Interpretation model sensitivity studies

3. Essential information for N-tipping and R-tipping
Numerical Bifurcation Theory

System of PDEs:

\[ \mathcal{M} \frac{\partial u}{\partial t} + \mathcal{L} u + \mathcal{N}(u) = \mathcal{F} \]

Operators containing parameters

Discretization (N)

Dynamical system:

\[ \mathcal{M}_N \frac{dx}{dt} + \mathcal{L}_N x + \mathcal{N}_N(x) = \mathcal{F}_N \]

x: state vector, dimension d
Autonomous systems: fixed points

$$\Phi(x, \lambda) = \mathcal{L}_N x + \mathcal{N}_N(x) - \mathcal{F}_N = 0$$

Arclength parametrization
Euler-Newton continuation

Starting Point:

\((x_0, \lambda_0)\)

Compute initial tangent:

\(\dot{\gamma}(s_0) = (\dot{x}(s_0), \dot{\lambda}(s_0))^T\)

Solve Extended system:

\(\Phi(\gamma(s)) = 0\)

\[\Sigma(x, \lambda, s) = \dot{x}_0^T(x - x_0) + \dot{\lambda}_0(\lambda - \lambda_0) - (s - s_0)\]

With Euler guess:

\[x^1 = x_0 + \Delta s \quad \dot{x}_0\]
\[\lambda^1 = \lambda_0 + \Delta s \quad \dot{\lambda}_0\]
The initial tangent

Differentiate: \[ \Phi(\gamma(s)) = 0 \] to \( s \):

\[
\begin{bmatrix}
\Phi_x & \Phi_\lambda
\end{bmatrix}
\gamma'(s) =
\begin{pmatrix}
\frac{\partial \Phi_1}{\partial x_1} & \cdots & \frac{\partial \Phi_1}{\partial x_N} \\
\frac{\partial \Phi_1}{\partial \Phi_1} & \cdots & \frac{\partial \Phi_1}{\partial \Phi_N} \\
\frac{\partial \Phi_1}{\partial x_N} & \cdots & \frac{\partial \Phi_1}{\partial \lambda}
\end{pmatrix}
\gamma'(s) = 0
\]

If \( (x_0, \lambda_0) \) is not a bifurcation point, then this matrix has rank \( N \)

First, the matrix \([\Phi_x \Phi_\lambda]\) is triangulated into the form

\[
\begin{pmatrix}
* & * & * & *
\\
0 & * & * & *
\\
0 & 0 & * & *
\end{pmatrix}
\]

\( v = (\dot{x}_0, \dot{\lambda}_0) \) can be computed by solving

\[
\begin{pmatrix}
* & * & * & *
\\
0 & * & * & *
\\
0 & 0 & * & *
\\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]
The Newton - Raphson process

Scalar function: \( G(x) = 0 \)

\[
G(x) = 0 \Rightarrow G'(x^k) \Delta x^{k+1} = -G(x^k)
\]

\[
y = G'(x^k)x + b \quad \text{and hence} \quad G(x^k) = G'(x^k)x^k + b
\]

Then:
\[
0 = G'(x^k)(x^{k+1} - x^k) + G(x^k)
\]
The Newton-Raphson process for pseudo-arclength continuation

Solve Extended system:

\[ \Phi(\gamma(s)) = 0 \]

\[ \Sigma(x, \lambda, s) = \dot{x}_0^T(x - x_0) + \dot{\lambda}_0(\lambda - \lambda_0) - (s - s_0) \]

NR-process:

\[
\begin{pmatrix}
\Phi_x(x^k, \lambda^k) & \Phi_\lambda(x^k, \lambda^k) \\
\dot{x}_0^T & \dot{\lambda}_0
\end{pmatrix}
\begin{pmatrix}
\Delta x^{k+1} \\
\Delta \lambda^{k+1}
\end{pmatrix}
= \begin{pmatrix}
-\Phi(x^k, \lambda^k) \\
\Delta s - \dot{x}_0^T(x^k - x_0) - \dot{\lambda}_0(\lambda^k - \lambda_0)
\end{pmatrix}
\]
Detection of bifurcation points

1. Direct indicators $f(s)$

$$\det(\Phi_x(s)) = 0$$

Saddle Node: $\dot{\lambda} = 0$

2. Solve linear stability problem

Use secant iteration to determine precise value

$$s_{l+1} = s_l - f(s_l) \frac{s_l - s_{l-1}}{f(s_l) - f(s_{l-1})}$$

$s_0 = s_a$ ; $s_1 = s_b$
Branch switching: Orthogonal to tangent
Branch switching: use imperfections

\[ p_s \]

Homotopy parameter
Linear stability

Dynamical system:
\[ \mathcal{M}_N \frac{dx}{dt} + \mathcal{L}_N x + \mathcal{N}_N(x) = \mathcal{F}_N \]

The linear stability problem of a fixed point leads to a generalized eigenvalue problem

\[ Ax = \sigma Bx \]

Solution methods:
1. QZ
2. Jacobi-Davidson QZ
3. Arnoldi
4. Simultaneous Iteration
Linear stability

\[ Ax = \sigma Bx \]

\[ \sigma = \sigma_r + i\sigma_i \ ; \ x = \hat{x}_r + i\hat{x}_i \]

How to detect bifurcation points?

Transcritical, Saddle-node, Pitchfork:
A single real eigenvalue crosses the imaginary axis

Hopf:
A complex conjugated pair of eigenvalue crosses the imaginary axis

Periodic orbit near Hopf bifurcation?

\[ \Phi(t) = e^{\sigma_r t} (\hat{x}_r \cos \sigma_i t - \hat{x}_i \sin \sigma_i t) \]
d < 10 : Matcont, Content, PyDSTool, and many more

d < 100 : AUTO

http://indy.cs.concordia.ca/auto/

d < 10^7 : Specialized, tailored codes
(similar setup as AUTO, but with different solvers)
Why determine full bifurcation diagrams of autonomous models (B-tipping)?

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2. (Correct) Interpretation model sensitivity studies

3. Essential information for N-tipping and R-tipping
Bifurcation diagram global ocean model

Dijkstra and Weijer, JPO, (2005)
The Atlantic steady state salt budget

\[ \theta = \theta_n \]

\[ S_0 \int_{S_{\text{atl}}} (E - P) \, dx \, dy \approx \int_{\theta_s} vS \, dx \, dz - \int_{\theta_n} vS \, dx \, dz \]

\[ \theta = \theta_s \]

Split zonal mean

\[ v' = v - \langle v \rangle \]

\[ S' = S - \langle S \rangle \]

De Vries and Weber, GRL, 2005

Dijkstra, Tellus, (2007)
Split freshwater transports

\[ M_{ov}(\theta) = -\frac{\eta}{S_0} \int \langle v \rangle (\langle S \rangle - S_0) \, dz ; \quad M_{az}(\theta) = -\frac{\eta}{S_0} \int \langle v' S' \rangle \, dz. \]

\( \text{MOC} \) exports salt if \( M_{ov} > 0 \) and \( \theta = \theta_s \).

\( \text{MOC} \) exports fresh water if \( M_{ov} < 0 \).

Rahmstorf, Clim Dyn. (1996)  
Weijer et al., JPO, (1999)  
De Vries and Weber, GRL, 2005  
Dijkstra, Tellus, (2007)
An indicator for the ME regime

\[
\int_{S_{\text{atl}}} (E - P) dx dy \approx M_{ov}(\theta_s) - M_{ov}(\theta_n) + M_{az}(\theta_s) - M_{az}(\theta_n)
\]

\[
\Sigma(\theta_n, \theta_s) = M_{ov}(\theta_s) - M_{ov}(\theta_n)
\]
northern North Atlantic freshwater input

Atlantic becomes fresher

MOC decrease

$\Sigma < 0$

MOC exports freshwater

less fresh water exported

More detailed physics:

When the spatial pattern of the MOC perturbation is similar to that of the background MOC, one can relate growth of perturbations to $\Sigma$

Huisman et al., JPO, (2010)
Analysis of multi-model GCMs

CMIP2 (IPCC-TAR)

Most models have a bias in Atlantic E - P!

CMIP3 (IPCC-AR4)

Weijer et al., JPO, (1999)

Drijfhout et al., (2010)

Drijfhout et al., Clim. Dyn., (2011)
Estimates from observations & reanalyses

Based on this indicator: Atlantic MOC is in a ME regime!

Hawkins et al., GRL, (2011)
Why determine full bifurcation diagrams of autonomous models (B-tipping)?

1. Physical mechanisms

2. (Correct) Interpretation model sensitivity studies

3. Essential information for N-tipping and R-tipping
Sensitivity to second parameter

\[ \lambda \]

\[ x \]

\[ \mu = \mu_0 \]

\[ \mu = \mu_1 \]

\[ \mu = \mu_2 \]
Extreme continuation: Bathymetry


1: Find $x^0$ satisfying $\|F(x^0, k^0)\|_2 < \epsilon$.
2: for $j = 1, 2, \ldots, p - 1$ do
3: Compute predictor $\mu(x^{j-1})$ based on difference $k^j - k^{j-1}$.
4: Let $G^j(x, \delta) = \cos^2 \theta M(x - \mu(x^{j-1})) + \sin^2 \theta F(x, k^j)$. Perform a pseudo-arclength continuation: $\delta = 0 \rightarrow \delta = 1$.
5: Store $x^j$ satisfying $\|G^j(x^j, 1)\|_2 = \|F(x^j, k^j)\|_2 < \epsilon$
6: end for
Results (Ocean GCM), 65M to 0M, idealized forcing